Rademacher functions in Morrey spaces

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Abstract

The Rademacher sums are investigated in the Morrey spaces $M_{p,w}$ on $[0,1]$ for $1 \le p < \infty$ and weight w being a quasi-concave function. They span l_2 space in $M_{p,w}$ if and only if the weight w is smaller than $\log_2^{-1/2} \frac{2}{t}$ on $(0,1)$. Moreover, if $1 < p < \infty$ the Rademacher sunspace $\mathcal{R}_p$ is complemented in $M_{p,w}$ if and only if it is isomorphic to l_2. However, the Rademacher subspace $\mathcal{R}_1$ is not complemented in $M_{1,w}$ for any quasi-concave weight w. In the last part of the paper geometric structure of Rademacher subspaces in Morrey spaces $M_{p,w}$ is described. It turns out that for any infinite-dimensional subspace X of $\mathcal{R}_p$ the following alternative holds: either X is isomorphic to l_2 or X contains a subspace which is isomorphic to c_0 and is complemented in $\mathcal{R}_p$.

1 Introduction and preliminaries

The well-known Morrey spaces introduced by Morrey in 1938 [20] in relation to the study of partial differential equations were widely investigated during last decades, including the study of classical operators of harmonic analysis: maximal, singular and potential operators – in various generalizations of these spaces. In the theory of partial differential equations, along with the weighted Lebesgue spaces, Morrey-type spaces also play an important role. They appeared to be quite useful in the study of the local behavior of the solutions of partial differential equations, a priori estimates and other topics.

Let $0 < p < \infty$, w be a non-negative non-decreasing function on $[0,\infty)$, and Ω a domain in $\mathbb{R}^n$. The *Morrey space* $M_{p,w} = M_{p,w}(\Omega)$ is the class of Lebesgue measurable real functions f on Ω such that

$$\|f\|_{M_{p,w}} = \sup_{0<r<\operatorname{diam}(\Omega),\, x_0\in\Omega} w(r) \left(\frac{1}{r} \int_{B_r(x_0)\cap\Omega} |f(t)|^p \, dt \right)^{1/p} < \infty, \tag{1}$$

where $B_r(x_0)$ is a ball with the center at x_0 and radius r. It is a quasi-Banach ideal space on Ω. The so-called ideal property means that if $|f| \le |g|$ a.e. on Ω and $g \in M_{p,w}$, then $f \in M_{p,w}$ and $\|f\|_{M_{p,w}} \le \|g\|_{M_{p,w}}$. In particular, if $w(r) = 1$ then $M_{p,w}(\Omega) = L_\infty(\Omega)$, if $w(r) = r^{1/p}$ then $M_{p,w}(\Omega) = L_p(\Omega)$ and in the case when $w(r) = r^{1/q}$ with $0 < p \le q < \infty$ $M_{p,w}(\Omega)$ are the classical Morrey spaces, denoted shortly by $M_{p,q}(\Omega)$ (see [14, Part 4.3],

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[15], [23] and [29]). Moreover, as a consequence of the Hölder-Rogers inequality we obtain monotonicity with respect to p, that is,

$$M_{p_1,w}(\Omega) \overset{1}{\hookrightarrow} M_{p_0,w}(\Omega) \ \text{ if } \ 0 < p_0 \le p_1 < \infty.$$

For two quasi-Banach spaces X and Y the symbol $X \overset{C}{\hookrightarrow} Y$ means that the embedding $X \subset Y$ is continuous and $\|f\|_Y \le C\|f\|_X$ for all $f \in X$.

It is easy to see that in the case when $\Omega = [0,1]$ quasi-norm (1) can be defined as follows

$$\|f\|_{M_{p,w}} = \sup_I \ w(|I|) \left(\frac{1}{|I|} \int_I |f(t)|^p \, dt \right)^{1/p}, \tag{2}$$

where the supremum is taken over all intervals I in $[0,1]$. In what follows $|E|$ is the Lebesgue measure of a set $E \subset \mathbb{R}$.

The main purpose of this paper is the investigation of the behaviour of Rademacher sums

$$R_n(t) = \sum_{k=1}^{n} a_k r_k(t), \ \ a_k \in \mathbb{R} \text{ for } k = 1, 2, ..., n, \ \text{ and } n \in \mathbb{N}$$

in general Morrey spaces $M_{p,w}$. Recall that the Rademacher functions on $[0,1]$ are defined by $r_k(t) = \operatorname{sign}(\sin 2^k \pi t), \ k \in \mathbb{N}, t \in [0,1]$.

The most important tool in studying Rademacher sums in the classical L_p-spaces and in general rearrangement invariant spaces is the so-called *Khintchine inequality* (cf. [11, p. 10], [1, p. 133], [16, p. 66] and [4, p. 743]): if $0 < p < \infty$, then there exist constants $A_p, B_p > 0$ such that for any sequence of real numbers $\{a_k\}_{k=1}^n$ and any $n \in \mathbb{N}$ we have

$$A_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2} \le \|R_n\|_{L_p[0,1]} \le B_p \Big(\sum_{k=1}^{n} |a_k|^2 \Big)^{1/2}. \tag{3}$$

Therefore, for any $1 \le p < \infty$, the Rademacher functions span in L_p an isomorphic copy of l_2. Also, the subspace $[r_n]$ is complemented in L_p for $1 < p < \infty$ and is not complemented in L_1 since no complemented infinite dimensional subspace of L_1 can be reflexive. In L_∞, the Rademacher functions span an isometric copy of l_1, which is uncomplemented.

The only non-trivial estimate for Rademacher sums in a general rearrangement invariant (r.i.) space X on $[0,1]$ is the inequality

$$\|R_n\|_X \le C \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2}, \tag{4}$$

where a constant $C > 0$ depends only on X. The reverse inequality to (4) is always true because $X \subset L_1$ and we can apply the left-hand side inequality from (3) for L_1. Paley and Zygmund [22] proved already in 1930 that estimate (4) holds for $X = G$, where G is the closure of $L_\infty[0,1]$ in the Orlicz space $L_M[0,1]$ generated by the function $M(u) = e^{u^2} - 1$. The proof can be found in Zygmund's classical books [30, p. 134] and [31, p. 214].

Later on Rodin and Semenov [25] showed that estimate (4) holds if and only if $G \subset X$. This inclusion means that X in a certain sense "lies far" from $L_\infty[0,1]$. In particular, G is

contained in every $L_p[0,1]$ for $p < \infty$. Moreover, Rodin-Semenov [26] and Lindenstrauss-Tzafriri [17, pp. 134-138] proved that $[r_n]$ is complemented in X if and only if $G \subset X \subset G'$, where G' denotes the Köthe dual space to G.

In contrast, Astashkin [3] studied the Rademacher sums in r.i. spaces which are situated very "close" to L_∞. In such a case a rather precise description of their behaviour may be obtained by using the real method of interpolation (cf. [10]). Namely, every space X that is interpolation between the spaces L_∞ and G can be represented in the form $X = (L_\infty, G)^K_\Phi$, for some parameter Φ of the real interpolation method, and then $\| \sum_{k=1}^\infty a_k r_k\|_X \approx \|\{a_k\}_{k=1}^\infty\|_F$, where $F = (l_1, l_2)^K_\Phi$.

Investigations of Rademacher sums in r.i. spaces are well presented in the books by Lindenstrauss-Tzafriri [17], Krein-Petunin-Semenov [13] and Astashkin [4]. At the same time, a very few papers are devoted to considering Rademacher functions in Banach function spaces, which are not r.i. Recently, Astashkin-Maligranda [6] initiated studying the behaviour of Rademacher sums in a weighted Korenblyum-Kreĭn-Levin space $K_{p,w}$, for $0 < p < \infty$ and a quasi-concave function w on $[0,1]$, equipped with the quasi-norm

$$\|f\|_{K_{p,w}} = \sup_{0<x\leq 1} w(x) \left(\frac{1}{x}\int_0^x |f(t)|^p \, dt\right)^{1/p} \tag{5}$$

(cf. [12], [18], [28, pp. 469-470], where $w(x) = 1$). If the supremum in (2) is taken over all subsets of $[0,1]$ of measure x, then we obtain an r.i. counterpart of the spaces $M_{p,w}$ and $K_{p,w}$, the Marcinkiewicz space $M^{(*)}_{p,w}[0,1]$, with the quasi-norm

$$\|f\|_{M^{(*)}_{p,w}} = \sup_{0<x\leq 1} w(x) \left(\frac{1}{x}\int_0^x f^*(t)^p \, dt\right)^{1/p}, \tag{6}$$

where f^* denotes the non-increasing rearrangement of $|f|$.

In what follows we consider only function spaces on $[0,1]$. Therefore, the weight w will be a non-negative non-decreasing function on $[0,1]$ and without loss of generality we will assume in the rest of the paper that $w(1) = 1$. Then, we have

$$L_\infty \overset{1}{\hookrightarrow} M^{(*)}_{p,w} \overset{1}{\hookrightarrow} M_{p,w} \overset{1}{\hookrightarrow} K_{p,w} \overset{1}{\hookrightarrow} L_p \tag{7}$$

because the corresponding suprema in (5), (2) and (6) are taken over larger classes of subsets of $[0,1]$.

Observe that if $\lim_{t\to 0^+} w(t) > 0$, then $M_{p,w} = M^{(*)}_{p,w} = L_\infty$, and if $\sup_{0<t\leq 1} w(t)\, t^{-1/p} < \infty$, then $M_{p,w} = L_p$ with equivalent quasi-norms. However, under appropriate assumptions on a weight w the second and the third inclusions in (7) are proper.

Proposition 1. *(i) If $\lim_{t\to 0^+} w(t)\, t^{-1/p} = \infty$, then there exists $f \in K_{p,w} \setminus M_{p,w}$.*

(ii) If $w(t)\, t^{-1/p}$ is a non-increasing function on $(0,1]$ *and* $\lim_{t\to 0^+} w(t) = \lim_{t\to 0^+} \frac{t^{1/p}}{w(t)} = 0$, *then there exists $g \in M_{p,w} \setminus M^{(*)}_{p,w}$.*

Proof. (i) Since $\lim_{t\to 0^+} w(t) t^{-1/p} = \infty$, there exists a sequence $\{t_k\} \subset (0,1]$ such that $t_k \searrow 0, t_1 \leq 1/2$ and $w(t_k) t_k^{-1/p} \nearrow \infty$. Let us denote $v(t) = w(t)\, t^{-1/p}$ and

$$g(s) := \sum_{k=1}^\infty \left(v(t_k)^{-p/2} - v(t_{k+1})^{-p/2}\right)^{1/p} (t_k - t_{k+1})^{-1/p} \chi_{(t_{k+1}, t_k]}(s).$$

Note that, by definition, $\operatorname{supp} g \subset [0,1/2]$. Then, for every $k \in \mathbb{N}$

$$\begin{aligned}\int_0^{t_k} |g(s)|^p\,ds &= \sum_{i=k}^{\infty}\int_{t_{i+1}}^{t_i} g(s)^p\,ds \\ &= \sum_{i=k}^{\infty}\frac{v(t_i)^{-p/2}-v(t_{i+1})^{-p/2}}{t_i-t_{i+1}}(t_i-t_{i+1}) = v(t_k)^{-p/2}.\end{aligned}$$

In particular, we see that $g \in L_p$. Let $f(t) := g(t+\frac{1}{2})$ for $0 \le t \le 1$. Then $\|f\|_p = \|g\|_p$, and therefore $f \in L_p$. Moreover, since $\operatorname{supp} f \subset [1/2,1]$, we obtain $f \in K_{p,w}$. In fact,

$$\begin{aligned}\|f\|_{K_{p,w}} &= \sup_{0<x\le 1} w(x)\left(\frac{1}{x}\int_0^x |f(t)|^p\,dt\right)^{1/p} = \sup_{\frac{1}{2}\le x\le 1}\frac{w(x)}{x^{1/p}}\Big(\int_{1/2}^x |f(t)|^p\,dt\Big)^{1/p} \\ &\approx \sup_{\frac{1}{2}\le x\le 1}\Big(\int_{1/2}^x |f(t)|^p\,dt\Big)^{1/p} = \|f\|_{L_p} < \infty.\end{aligned}$$

At the same time, if $I_k := [\frac{1}{2}, t_k+\frac{1}{2}]$, $k = 1,2,\ldots$, we have

$$w(|I_k|)\left(\frac{1}{|I_k|}\int_{I_k}|f(t)|^p\,dt\right)^{1/p} = v(t_k)\Big(\int_0^{t_k}|g(s)|^p\,ds\Big)^{1/p} = v(t_k)\cdot v(t_k)^{-1/2} = v(t_k)^{1/2}.$$

Since $v(t_k) \nearrow \infty$ as $k \to \infty$, we conclude that $f \notin M_{p,w}$.

(ii) It is easy to find a function $g \in L_p \setminus M_{p,w}^{(*)}$. Next, by the main result of the paper [2], there exist a function $f \in M_{p,w}$ and constants $c_0 > 0$ and $\lambda_0 > 0$ such that

$$\Big|\{t\in[0,1] : |f(t)| > \lambda\}\Big| \ge c\,\Big|\{t\in[0,1] : |g(t)| > \lambda\}\Big|$$

for all $\lambda \ge \lambda_0$. Clearly, since $g \notin M_{p,w}^{(*)}$, from the last inequality it follows that $f \notin M_{p,w}^{(*)}$. $\square$

The proof of Proposition 1 (ii) shows also that the Morrey space $M_{p,w}$ is not an r.i. space whenever $w(t)\,t^{-1/p}$ is a non-increasing function on $(0,1]$ and $\lim_{t\to 0^+} w(t) = \lim_{t\to 0^+}\frac{t^{1/p}}{w(t)} = 0$.

For a normed ideal space $X = (X, \|\cdot\|)$ on $[0,1]$ the *Köthe dual* (or *associated space*) X' is the space of all real-valued Lebesgue measurable functions defined on $[0,1]$ such that the *associated norm*

$$\|f\|_{X'} := \sup_{g\in X,\ \|g\|_X\le 1}\int_0^1 |f(x)g(x)|\,dx$$

is finite. The Köthe dual X' is a Banach ideal space. Moreover, $X \overset{1}{\hookrightarrow} X''$ and we have equality $X = X''$ with $\|f\| = \|f\|_{X''}$ if and only if the norm in X has the *Fatou property*, that is, if $0 \le f_n \nearrow f$ a.e. on $[0,1]$ and $\sup_{n\in\mathbf{N}}\|f_n\| < \infty$, then $f \in X$ and $\|f_n\| \nearrow \|f\|$.

Denote by $\mathcal{D}$ the set of all dyadic intervals $I_k^n = [(k-1)\,2^{-n}, k\,2^{-n}]$, where $n = 0,1,2,\ldots$ and $k = 1,2,\ldots,2^n$. If f and g are nonnegative functions (or quasi-norms), then the symbol $f \approx g$ means that $C^{-1}g \le f \le C\,g$ for some $C \ge 1$. Moreover, we write $X \simeq Y$ if Banach spaces X and Y are isomorphic.

The paper is organized as follows. After Introduction, in Section 2 the behaviour of Rademacher sums in Morrey spaces is described (see Theorem 1). The main result of Section 3 is Theorem 2, which states that the Rademacher subspace $\mathcal{R}_p, 1 < p < \infty$, is complemented in the Morrey space $M_{p,w}$ if and only if $\mathcal{R}_p$ is isomorphic to l_2 or equivalently if $\sup_{0<t\le 1} w(t)\log_2^{1/2}(2/t) < \infty$. In the case when $p = 1$ situation is different, which is the contents of Section 4, where we are proving in Theorem 3 that the subspace $\mathcal{R}_1$ is not complemented in $M_{1,w}$ for any quasi-concave weight w. Finally, in Section 5, the geometric structure of Rademacher subspaces in Morrey spaces is investigated (see Theorem 4).

2 Rademacher sums in Morrey spaces

We start with the description of behaviour of Rademacher sums in the Morrey spaces $M_{p,w}$ defined by quasi-norms (2), where $0 < p < \infty$ and w is a non-decreasing function on $[0,1]$ satisfying the doubling condition $w(2t) \le C_0\, w(t)$ for all $t \in (0, 1/2]$ with a certain $C_0 \ge 1$.

THEOREM 1. *With constants depending only on p and w*

$$\Big\| \sum_{k=1}^{\infty} a_k\, r_k \Big\|_{M_{p,w}} \approx \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} \Big(w(2^{-m}) \sum_{k=1}^{m} |a_k| \Big). \tag{8}$$

Proof. Firstly, let $1 \le p < \infty$. Consider an arbitrary interval $I \in \mathcal{D}$, i.e., $I = I_k^m$, with $m \in \mathbb{N}$ and $k = 1, 2, \ldots, 2^m$. Then, for every $f = \sum_{k=1}^{\infty} a_k\, r_k$, we have

$$\Big(\int_I |f(t)|^p\, dt \Big)^{1/p} = \Big(\int_I \Big| \sum_{k=1}^{m} a_k \varepsilon_k + \sum_{k=m+1}^{\infty} a_k r_k(t) \Big|^p dt \Big)^{1/p},$$

where $\varepsilon_k = \operatorname{sign} r_k\big|_I$, $k = 1, 2, \ldots, m$. Since the functions

$$\sum_{k=1}^{m} a_k \varepsilon_k + \sum_{k=m+1}^{\infty} a_k r_k(t) \quad \text{and} \quad \sum_{k=1}^{m} a_k \varepsilon_k - \sum_{k=m+1}^{\infty} a_k r_k(t)$$

are equimeasurable on the interval I, it follows that

$$\begin{aligned} \Big(\int_I |f(t)|^p\, dt \Big)^{1/p} &= \frac{1}{2} \Big(\int_I \Big| \sum_{k=1}^{m} a_k \varepsilon_k + \sum_{k=m+1}^{\infty} a_k r_k(t) \Big|^p dt \Big)^{1/p} \\ &+ \frac{1}{2} \Big(\int_I \Big| \sum_{k=1}^{m} a_k \varepsilon_k - \sum_{k=m+1}^{\infty} a_k r_k(t) \Big|^p dt \Big)^{1/p}, \end{aligned}$$

whence by the Minkowski triangle inequality we obtain

$$\Big(\int_I |f(t)|^p\, dt \Big)^{1/p} \ge \Big(\int_I \Big| \sum_{k=1}^{m} a_k \varepsilon_k \Big|^p dt \Big)^{1/p} = 2^{-m/p} \Big| \sum_{k=1}^{m} a_k \varepsilon_k \Big|$$

for every $m = 1, 2, \ldots$. Clearly, one may find $i = 1, 2, \ldots, 2^m$ such that $r_k\big|_{I_i^m} = \operatorname{sign} a_k$, for all $k = 1, 2, \ldots, m$. Therefore, for every $m = 1, 2, \ldots$

$$\left(\frac{1}{|I|}\int_I |f(t)|^p\,dt\right)^{1/p} \geq \sum_{k=1}^m |a_k|,$$

and so

$$\|f\|_{M_{p,w}} \geq \sup_{m\in\mathbb{N}} w(2^{-m}) \sum_{k=1}^m |a_k|.$$

On the other hand, by (7) and (3) we have

$$\|f\|_{M_{p,w}} \geq \|f\|_{L_p} \geq A_p\,\|\{a_k\}_{k=1}^\infty\|_{l_2}.$$

Combining these inequalities, we obtain

$$\|f\|_{M_{p,w}} \geq \frac{A_p}{2}\Big(\|\{a_k\}_{k=1}^\infty\|_{l_2} + \sup_{m\in\mathbb{N}} w(2^{-m}) \sum_{k=1}^m |a_k|\Big).$$

Let us prove the reverse inequality. For a given interval $I \subset [0,1]$ we can find two adjacent dyadic intervals I_1 and I_2 of the same length such that

$$I \subset I_1 \cup I_2 \ \text{ and } \ \frac{1}{2}\,|I_1| \leq |I| \leq 2\,|I_1|. \tag{9}$$

If $|I_1| = |I_2| = 2^{-m}$, then by the Minkowski triangle inequality and inequality in (3) we have

$$\begin{aligned}
\Big(\int_{I_1} |f(t)|^p\,dt\Big)^{1/p} &= \Big(\int_{I_1} \Big|\sum_{k=1}^m a_k\varepsilon_k + \sum_{k=m+1}^\infty a_k r_k(t)\Big|^p\,dt\Big)^{1/p}\\
&\leq \Big(\int_{I_1} \Big|\sum_{k=1}^m a_k\varepsilon_k\Big|^p\,dt\Big)^{1/p} + \Big(\int_{I_1} \Big|\sum_{k=m+1}^\infty a_k r_k(t)\Big|^p\,dt\Big)^{1/p}\\
&\leq 2^{-m/p}\sum_{k=1}^m |a_k| + 2^{-m/p}\Big(\int_0^1 \Big|\sum_{k=m+1}^\infty a_k r_{k-m}(t)\Big|^p\,dt\Big)^{1/p}\\
&\leq 2^{-m/p}\sum_{k=1}^m |a_k| + 2^{-m/p}\,B_p\,\|\{a_k\}_{k=1}^\infty\|_{l_2}.
\end{aligned}$$

The same estimate holds also for the integral $\big(\int_{I_2} |f(t)|^p\,dt\big)^{1/p}$. Therefore, by (9),

$$\begin{aligned}
\Big(\frac{1}{|I|}\int_I |f(t)|^p\,dt\Big)^{1/p} &\leq 2^{1/p}\Big(\frac{1}{|I_1|}\int_{I_1} |f(t)|^p\,dt + \frac{1}{|I_2|}\int_{I_2} |f(t)|^p\,dt\Big)^{1/p}\\
&\leq 4^{1/p}\,B_p\Big(\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\Big)
\end{aligned}$$

and

$$
\begin{aligned}
w(|I|)\left(\frac{1}{|I|}\int_I |f(t)|^p\,dt\right)^{1/p} &\le w(2\cdot 2^{-m})\,4^{1/p}\,B_p\left(\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\right)\\
&\le C_0\cdot 4^{1/p}\,B_p\,w(2^{-m})\left(\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\right).
\end{aligned}
$$

Hence, using definition of the norm in $M_{p,w}$, we obtain

$$
\|f\|_{M_{p,w}} \le C_0\cdot 4^{1/p}\,B_p\left(\sup_{m\in\mathbb{N}} w(2^{-m})\sum_{k=1}^m |a_k| + \|\{a_k\}_{k=1}^\infty\|_{l_2}\right).
$$

The same proof works also in the case when $0 < p < 1$ with the only change that the L_p-triangle inequality contains constant $2^{1/p-1}$. $\square$

In the rest of the paper, a weight function w is assumed to be *quasi-concave on* $[0,1]$, that is, $w(0)=0, w$ is non-decreasing, and $w(t)/t$ is non-increasing on $(0,1]$. Moreover, as above, we assume that $w(1)=1$.

Recall that a basic sequence $\{x_k\}$ in a Banach space X is called *subsymmetric* if it is unconditional and is equivalent in X to any its subsequence.

Corollary 1. *For every $1\le p<\infty$ $\{r_k\}$ is an unconditional and not subsymmetric basic sequence in $M_{p,w}$.*

Corollary 2. *Let $1\le p<\infty$. The Rademacher functions span l_2 space in $M_{p,w}$ if and only if*

$$
\sup_{0<t\le 1} w(t)\log_2^{1/2}(2/t) < \infty. \tag{10}
$$

Proof. If (10) holds, then for all $m\in\mathbb{N}$ we have $w(2^{-m})\,m^{1/2}\le C$. Using the Hölder-Rogers inequality, we obtain

$$
w(2^{-m})\sum_{k=1}^m |a_k| \le w(2^{-m})\left(\sum_{k=1}^m |a_k|^2\right)^{1/2} m^{1/2} \le C\left(\sum_{k=1}^m |a_k|^2\right)^{1/2}.
$$

Therefore, from (8) it follows that $\|\sum_{k=1}^\infty a_k\,r_k\|_{M_{p,w}} \approx \|\{a_k\}\|_{l_2}$.

Conversely, suppose that condition (10) does not hold. Then, by the quasi-concavity of w, there exists a sequence of natural numbers $m_k\to\infty$ such that

$$
w(2^{-m_k})\,m_k^{1/2}\to\infty \text{ as } k\to\infty. \tag{11}
$$

Consider the Rademacher sums $R_k(t)=\sum_{i=1}^k a_i^k\,r_i(t)$ corresponding to the sequences of coefficients $a^k=(a_i^k)_{i=1}^{m_k}$, where $a_i^k = m_k^{-1/2}$, $1\le i\le m_k$. We have $\|a^k\|_{l_2}=1$ for all $k=1,2,\ldots$ However, $\sum_{i=1}^{m_k} a_i^k = m_k^{1/2}$ $(k=1,2,\ldots)$, which together with (11) and (8) imply that $\|R_k\|_{M_{p,w}}\to\infty$ as $k\to\infty$. $\square$

Remark 1. The Rademacher functions span l_2 in each of the spaces $M_{p,w}^{(*)}$, $M_{p,w}$ and $K_{p,w}$, $1 \le p < \infty$ (see embeddings (7)). In fact, the Orlicz space L_M generated by the function $M(u) = e^{u^2} - 1$ coincides with the Marcinkiewicz space $M_{1,v}^{(*)}$ with $v(t) = \log_2^{-1/2}(2/t)$ (cf. [4, Lemma 3.2]). Recalling that G is the closure of L_∞ in $M_{1,v}^{(*)}$ we note that the embedding $G \subset M_{p,w}^{(*)}$ holds if and only if (10) is satisfied. Therefore, by already mentioned Rodin-Semenov theorem (cf. [25]; see also [17, Theorem 2.b.4]), the Rademacher functions span l_2 in $M_{p,w}^{(*)}$ if and only if (10) holds.

Moreover, it is instructive to compare the behaviour of Rademacher sums in the spaces $M_{1,w}^{(*)}$, $M_{1,w}$ and $K_{1,w}$ in the case when $w(t) = \log_2^{-1/q}(2/t)$, where $q > 2$. Then (10) does not hold and

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{M_{1,w}^{(*)}} \asymp \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} m^{-1/q} \sum_{k=1}^{m} a_k^*,$$

where $\{a_k^*\}$ is the non-increasing rearrangement of $\{|a_k|\}_{k=1}^{\infty}$ (cf. Rodin-Semenov [25, p. 221] and Pisier [24]; see also Marcus-Pisier [19, pp. 277-278]),

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{M_{1,w}} \asymp \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} m^{-1/q} \sum_{k=1}^{m} |a_k| \quad \text{by } (8), \text{ and}$$

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{K_{1,w}} \asymp \|\{a_k\}_{k=1}^{\infty}\|_{l_2} + \sup_{m\in\mathbb{N}} m^{-1/q} \Big| \sum_{k=1}^{m} a_k \Big| \quad \text{(cf. [6, Theorem 2]).}$$

Now, we pass to studying the problem of complementability of the closed linear span $\mathcal{R}_p := [r_n]_{n=1}^{\infty}$ in the space $M_{p,w}$. Since the results turn out to be different for $p > 1$ and $p = 1$, we consider these cases separately.

3 Complementability of Rademacher subspaces in Morrey spaces $M_{p,w}$ for $p > 1$

THEOREM 2. *Let $1 < p < \infty$. The subspace $\mathcal{R}_p$ is complemented in the Morrey space $M_{p,w}$ if and only if condition (10) holds.*

To prove this theorem we will need the following auxiliary assertion.

Proposition 2. *If condition (10) does not hold, then the subspace $\mathcal{R}_p$ contains a complemented (in $\mathcal{R}_p$) subspace isomorphic to c_0.*

Proof. Since w is quasi-concave, by the assumption, we have

$$\limsup_{n\to\infty} w(2^{-n})\sqrt{n} = \infty. \tag{12}$$

We select an increasing sequence of positive integers as follows. Let n_1 be the least positive integer satisfying the inequality $w(2^{-n_1})\sqrt{n_1} \ge 2$. As it is easy to see $w(2^{-n_1})\sqrt{n_1} < 2^2$. By

induction, assume that the numbers $n_1 < n_2 < \ldots < n_{k-1}$ are chosen. Applying (12), we take for n_k the least positive integer such that

$$w(2^{-n_k})\sqrt{n_k - n_{k-1}} \geq 2^k. \tag{13}$$

Then, obviously,

$$w(2^{-n_k})\sqrt{n_k - n_{k-1}} < 2^{k+1}. \tag{14}$$

Thus, we obtain a sequence $0 = n_0 < n_1 < \ldots$ satisfying inequalities (13) and (14) for all $k \in \mathbb{N}$. Let us consider the block basis $\{v_k\}_{k=1}^{\infty}$ of the Rademacher system defined as follows:

$$v_k = \sum_{i=n_{k-1}+1}^{n_k} a_i\, r_i, \quad \text{where} \quad a_i = \frac{1}{(n_k - n_{k-1})\, w(2^{-n_k})} \quad \text{for} \quad n_{k-1} < i \leq n_k.$$

Let us recall that, by Theorem 1, if $R = \sum_{k=1}^{\infty} b_k\, r_k$, then $\|R\|_{M_{p,w}} \approx \|R\|_{l_2} + \|R\|_w$, where

$$\|R\|_{l_2} = \Big(\sum_{k=1}^{\infty} b_k^2\Big)^{1/2} \quad \text{and} \quad \|R\|_w = \sup_{m \in \mathbb{N}} \Big(w(2^{-m}) \sum_{k=1}^{m} |b_k|\Big).$$

Now, we estimate the norm of v_k, $k = 1, 2, \ldots$, in $M_{p,w}$. At first, by (13),

$$\|v_k\|_{l_2} = \Big(\sum_{i=n_{k-1}+1}^{n_k} a_i^2\Big)^{1/2} = \frac{1}{\sqrt{n_k - n_{k-1}}\, w(2^{-n_k})} \leq 2^{-k}, \; k = 1, 2, \ldots \tag{15}$$

Moreover, taking into account (13), (14) and the choice of n_k, for every $k \in \mathbb{N}$ and $n_{k-1} < i \leq n_k$ we have

$$w(2^{-i}) \sum_{j=n_{k-1}+1}^{i} a_j = \frac{w(2^{-i})(i - n_{k-1})}{(n_k - n_{k-1})w(2^{-n_k})} \leq \frac{2^{k+1}\sqrt{i - n_{k-1}}}{2^k \sqrt{n_k - n_{k-1}}} \leq 2.$$

Therefore, $\|v_k\|_w \leq 2$ for $k \in \mathbb{N}$ and combining this with (15) we obtain $\|v_k\|_{M_{p,w}} \leq C$ for $k \in \mathbb{N}$.

On the other hand, by Theorem 1,

$$\|v_k\|_{M_{p,w}} \geq c\, w(2^{-n_k}) \sum_{i=n_{k-1}+1}^{n_k} a_i = c \tag{16}$$

for some constant $c > 0$ and every $k \in \mathbb{N}$. Thus, $\{v_k\}_{k=1}^{\infty}$ is a semi-normalized block basis of $\{r_k\}_{k=1}^{\infty}$ in $M_{p,w}$.

Further, let us select a subsequence $\{m_i\} \subset \{n_k\}$ such that

$$w(2^{-m_{i+1}}) \leq \frac{1}{2} w(2^{-m_i}), \quad i = 1, 2, \ldots \tag{17}$$

and denote by $\{u_i\}_{i=1}^{\infty}$ the corresponding subsequence of $\{v_k\}_{k=1}^{\infty}$. Then, u_i can be represented as follows:

$$u_i = \sum_{k=l_i}^{m_i} a_k\, r_k, \quad \text{where} \quad l_i = n_{j_i - 1} + 1, \; m_i = n_{j_i}, \; j_1 < j_2 < \ldots.$$

Moreover, from the above $\{u_i\}$ is a semi-normalized sequence in $M_{p,w}$ and

$$\|u_i\|_{l_2} \le 2^{-i} \quad \text{for} \quad i = 1, 2, \ldots. \tag{18}$$

We show that the sequence $\{u_i\}_{i=1}^{\infty}$ is equivalent in $M_{p,w}$ to the unit vector basis of c_0.

Let $f = \sum_{i=1}^{\infty} \beta_i u_i, \beta_i \in \mathbb{R}$. Then, we have

$$f = \sum_{i=1}^{\infty} \beta_i \sum_{k=l_i}^{m_i} a_k r_k = \sum_{k=1}^{\infty} \gamma_k r_k,$$

where $\gamma_k = \beta_i a_k, l_i \le k \le m_i, i = 1, 2, \ldots$ and $\gamma_k = 0$ if $k \notin \cup_{i=1}^{\infty}[l_i, m_i]$. To estimate $\|f\|_w$, assume, at first, that $m_s \le q < l_{s+1}$ for some $s \in \mathbb{N}$. Then,

$$\sum_{k=1}^{q} |\gamma_k| = \sum_{i=1}^{s} |\beta_i| \sum_{k=l_i}^{m_i} a_k = \sum_{i=1}^{s} |\beta_i| \frac{1}{w(2^{-m_i})} \le \|(\beta_i)\|_{c_0} \sum_{i=1}^{s} \frac{1}{w(2^{-m_i})},$$

and from (17) it follows that

$$w(2^{-q}) \sum_{k=1}^{q} |\gamma_k| \le \|(\beta_i)\|_{c_0} \sum_{i=1}^{s} \frac{w(2^{-m_s})}{w(2^{-m_i})} \le \|(\beta_i)\|_{c_0} \sum_{i=0}^{\infty} 2^{-i} = 2\,\|(\beta_i)\|_{c_0}.$$

Otherwise, we have $l_s \le q < m_s, s \in \mathbb{N}$. Then, similarly,

$$\begin{aligned}
\sum_{k=1}^{q} |\gamma_k| \quad &\le \quad \left(\sum_{i=1}^{s-1} \frac{1}{w(2^{-m_i})} + \sum_{k=l_s}^{q} a_k \right) \|(\beta_i)\|_{c_0} \\
&= \quad \left(\sum_{i=1}^{s-1} \frac{1}{w(2^{-m_i})} + \frac{q - l_s + 1}{(m_s - l_s + 1)\, w(2^{-m_s})} \right) \|(\beta_i)\|_{c_0}.
\end{aligned}$$

Since $m_s = n_{j_s}$ and $l_s = n_{j_s - 1} + 1$ for some $j_s \in \mathbb{N}$, in view of (13), (17) and the choice of n_{j_s}, we obtain

$$\begin{aligned}
w(2^{-q}) \sum_{k=1}^{q} |\gamma_k| \quad &\le \quad \left(\sum_{i=1}^{s-1} \frac{w(2^{-m_{s-1}})}{w(2^{-m_i})} + \frac{w(2^{-q})(q - l_s + 1)}{(m_s - l_s + 1)\, w(2^{-m_s})} \right) \|(\beta_i)\|_{c_0} \\
&\le \quad \left(\sum_{i=0}^{\infty} 2^{-i} + \frac{2^{j_s+1}\sqrt{q - l_s + 1}}{2^{j_s}\sqrt{m_s - l_s + 1}} \right) \|(\beta_i)\|_{c_0} \le 4\,\|(\beta_i)\|_{c_0}.
\end{aligned}$$

Combining this with the previous estimate, we obtain that $\|f\|_w \le 4\,\|(\beta_i)\|_{c_0}$. On the other hand, from (18) it follows that $\|f\|_{l_2} \le \|(\beta_i)\|_{c_0}$. Therefore, again by Theorem 1,

$$\|f\|_{M_{p,w}} \le C \left(\|f\|_{l_2} + \|f\|_w \right) \le 5\, C\, \|(\beta_i)\|_{c_0}.$$

In opposite direction, taking into account the fact that $\{u_i\}$ is an unconditional sequence in $M_{p,w}$, by (16), we obtain

$$\|f\|_{M_{p,w}} \ge c' \sup_{i \in \mathbb{N}} |\beta_i| \, \|u_i\|_{M_{p,w}} \ge c' c \, \|(\beta_i)\|_{c_0},$$

for some constant $c' > 0$. Thus, we have proved that $E := [u_n]_{M_{p,w}} \simeq c_0$. Since $\mathcal{R}_p$ is separable, Sobczyk's theorem (see, for example, [1, Corollary 2.5.9]) implies that E is a complemented subspace in $\mathcal{R}_p$. $\square$

Proof of Theorem 2. At first, let us assume that relation (10) holds. Then, by Corollary 2, $\mathcal{R}_p \simeq l_2$. Therefore, since $M_{p,w} \overset{1}{\hookrightarrow} L_p$, by the Khintchine inequality, the orthogonal projection P generated by the Rademacher system satisfies the following:

$$\|Pf\|_{M_{p,w}} \approx \|Pf\|_{L_p} \le \|P\|_{L_p \to L_p} \|f\|_{L_p} \le \|P\|_{L_p \to L_p} \|f\|_{M_{p,w}},$$

because P is bounded in $L_p, 1 < p < \infty$. Hence, $P : M_{p,w} \to M_{p,w}$ is bounded.

Conversely, we argue in a similar way as in the proof of Theorem 4 in [5]. Suppose that the subspace $\mathcal{R}_p = [r_n]_{n=1}^{\infty}$ is complemented in $M_{p,w}$ and let $P_1 : M_{p,w} \to M_{p,w}$ be a bounded linear projection whose range is $\mathcal{R}_p$. By Proposition 2, there is a subspace E complemented in $\mathcal{R}_p$ and such that $E \simeq c_0$. Let $P_2 : \mathcal{R}_p \to E$ be a bounded linear projection. Then $P := P_2 \circ P_1$ is a linear projection bounded in $M_{p,w}$ whose image coincides with E. Thus, $M_{p,w}$ contains a complemented subspace $E \simeq c_0$.

Since $M_{p,w}$ is a conjugate space (more precisely, $M_{p,w} = (H^{q,u})^*$, where $H^{q,u}$ is the "block space" and $1/p + 1/q = 1$ – see, for example, [29, Proposition 5]; see also [9] and [21]), this contradicts the well-known result due to Bessaga-Pełczyński saying that arbitrary conjugate space cannot contain a complemented subspace isomorphic to c_0 (see [8, Corollary 4] and [7, Theorem 4 and its proof]). This contradiction proves the theorem. $\square$

4 Rademacher subspace $\mathcal{R}_1$ is not complemented in Morrey space $M_{1,w}$

THEOREM 3. *For every quasi-concave weight w the subspace $\mathcal{R}_1$ is not complemented in the Morrey space $M_{1,w}$.*

In the proof we consider two cases separately, depending if the condition (10) is satisfied or not.

Proof of Theorem 3: the case when (10) does not hold. On the contrary, we suppose that $\mathcal{R}_1$ is complemented in $M_{1,w}$. Then, if Q is a bounded linear projection from $M_{1,w}$ onto $\mathcal{R}_1$, by Theorem 1, for every $p \in (1, \infty)$ and $f \in M_{p,w}$, we have

$$\|Qf\|_{M_{p,w}} \approx \|Qf\|_{M_{1,w}} \le \|Q\| \, \|f\|_{M_{1,w}} \le \|Q\| \, \|f\|_{M_{p,w}}.$$

Thus, Q is a bounded projection from $M_{p,w}$ onto $\mathcal{R}_p$, which contradicts Theorem 2. $\square$

To prove the assertion in the case when (10) holds, we will need auxiliary results. Let $M_{p,w}^d$ be the dyadic version of the space $M_{p,w}$, $1 \le p < \infty$, consisting of all measurable functions $f : [0,1] \to \mathbb{R}$ such that

$$\|f\|_{M_{p,w}^d} = \sup_{I \in \mathcal{D}} w(|I|) \left(\frac{1}{|I|} \int_I |f(t)|^p \, dt \right)^{1/p} < \infty.$$

Lemma 1. *For every $1 \le p < \infty$ $M_{p,w} = M_{p,w}^d$ and*

$$\|f\|_{M_{p,w}^d} \le \|f\|_{M_{p,w}} \le 4 \, \|f\|_{M_{p,w}^d}. \tag{19}$$

Proof. The left-hand side inequality in (19) is obvious. To prove the right-hand side one, we observe that for any interval $I \subset [0,1]$ we can find adjacent dyadic intervals I_1 and I_2 of the same length such that $I \subset I_1 \cup I_2$ and $\frac{1}{2}|I_1| \le |I| \le 2|I_1|$. Then, by the quasi-concavity of w,

$$\begin{aligned} w(|I|)\left(\frac{1}{|I|}\int_I |f(t)|^p\,dt\right)^{1/p} &= \frac{w(|I|)}{|I|}\Big(|I|^{p-1}\int_I |f(t)|^p\,dt\Big)^{1/p} \\ \le\ & \frac{w(\frac{1}{2}|I_1|)}{\frac{1}{2}|I_1|}\Big[2^{p-1}|I_1|^{p-1}\Big(\int_{I_1}|f(t)|^p\,dt + \int_{I_2}|f(t)|^p\,dt\Big)\Big]^{1/p} \\ \le\ & 2^{2-1/p}\,w(|I_1|)\left(\frac{1}{|I_1|}\int_{I_1}|f(t)|^p\,dt + \frac{1}{|I_2|}\int_{I_2}|f(t)|^p\,dt\right)^{1/p} \\ \le\ & 4\sup_{I\in\mathcal{D}} w(|I|)\left(\frac{1}{|I|}\int_I |f(t)|^p\,dt\right)^{1/p} = 4\,\|f\|_{M^d_{p,w}}. \end{aligned}$$

Taking the supremum over all intervals $I \subset [0,1]$, we obtain the right-hand side inequality in (19). □

Let P be the orthogonal projection generated by the Rademacher sequence, i.e.,

$$Pf(t) := \sum_{k=1}^{\infty}\int_0^1 f(s)r_k(s)\,ds\cdot r_k(t).$$

Proposition 3. *Let $1 \le p < \infty$. If $\mathcal{R}_p$ is a complemented subspace in $M_{p,w}$, then the projection P is bounded in $M_{p,w}$.*

Proof. By Lemma 1, it is sufficient to prove the same assertion for the dyadic space $M^d_{p,w}$. We almost repeat the arguments from the proof of the similar result for r.i. function spaces (see [26] or [4, Theorem 3.4]).

Let $t = \sum_{i=1}^{\infty}\alpha_i 2^{-i}$ and $u = \sum_{i=1}^{\infty}\beta_i 2^{-i}$ ($\alpha_i, \beta_i = 0, 1$) be the binary expansion of the numbers $t, u \in [0,1]$. Define the following operation:

$$t \oplus u = \sum_{i=1}^{\infty} 2^{-i}[(\alpha_i + \beta_i) \bmod 2].$$

One can easily verify that this operation transforms the segment $[0,1]$ into a compact Abelian group. For every $u \in [0,1]$, the transformation $w_u(s) = s \oplus u$ preserves the Lebesgue measure on $[0,1]$, i.e., for any measurable $E \subset [0,1]$, its inverse image $w_u^{-1}(E)$ is measurable and $m(w_u^{-1}(E)) = m(E)$. Moreover, w_u maps any dyadic interval onto some dyadic interval. Hence, the operators $T_u f = f \circ w_u$ $(0 \le u \le 1)$ act isometrically in $M^d_{p,w}$. From the definition of the Rademacher functions it follows that the subspace $\mathcal{R}_p$ is invariant with respect to these operators. Therefore, by the Rudin theorem (see [27, Theorem 5.18, pp. 134-135]), there exists a bounded linear projector Q acting from $M^d_{p,w}$ onto $\mathcal{R}_p$ and commuting with all operators T_u $(0 \le u \le 1)$. We show that $Q = P$.

First of all, the projector Q has the representation

$$Qf(t) = \sum_{i=1}^{\infty} Q_i(f)\, r_i(t), \tag{20}$$

where by Theorem 1, Q_i $(i = 1, 2, \ldots)$ are linear bounded functionals on $M^d_{p,w}$. It is obvious that

$$Q_i(r_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases} \tag{21}$$

Consider the sets

$$U_i = \Big\{ u \in [0,1] : u = \sum_{j=1}^{\infty} \alpha_j 2^{-j}, \;\; \alpha_i = 0 \Big\}, \quad U_i^c = [0,1] \backslash U_i.$$

One can check that

$$r_i(t \oplus u) = \begin{cases} \;\; r_i(t) \text{ if } u \in U_i, \\ -r_i(t) \;\text{ if } u \in U_i^c. \end{cases}$$

Due to the relation $T_u Q = Q T_u$ $(0 \leq u \leq 1)$ this implies

$$Q_i(T_u f) = \begin{cases} \;\; Q_i(f) \text{ if } u \in U_i, \\ -Q_i(f) \;\text{ if } u \in U_i^c. \end{cases}$$

Taking into account that $m(U_i) = m(U_i^c) = 1/2$, we find that

$$\int_{U_i} Q_i(T_u f)\, du = \frac{1}{2} Q_i(f) \;\text{ and }\; \int_{U_i^c} Q_i(T_u f)\, du = -\frac{1}{2} Q_i(f).$$

Thanks to the boundedness of Q_i , this functional can be moved outside the integral; therefore, we obtain

$$Q_i(f) = Q_i\Big(\int_{U_i} T_u f\, du - \int_{U_i^c} T_u f\, du \Big). \tag{22}$$

Since

$$\{s \in [0,1]: \; s = t \oplus u, \;\; u \in U_i\} = \begin{cases} \; U_i \text{ if } t \in U_i, \\ U_i^c \text{ if } t \in U_i^c, \end{cases}$$

$$\{s \in [0,1]: \; s = t \oplus u, \;\; u \in U_i^c\} = \begin{cases} U_i^c \text{ if } t \in U_i, \\ U_i \text{ if } t \in U_i^c, \end{cases}$$

and the transformation ω_u preserves the Lebesgue measure on $[0,1]$, we have

$$\int_{U_i} T_u f(t) du \; = \; \int_{U_i} f(s)\, ds \cdot \chi_{U_i}(t) \; + \; \int_{U_i^c} f(s)\, ds \cdot \chi_{U_i^c}(t)$$

and

$$\int_{U_i^c} T_u f(t)\, du \; = \; \int_{U_i^c} f(s)\, ds \cdot \chi_{U_i}(t) \; + \; \int_{U_i} f(s)\, ds \cdot \chi_{U_i^c}(t).$$

It is easy to see that $r_i(t) = \chi_{U_i}(t) - \chi_{U_i^c}(t)$. Therefore, from the last two relations it follows that

$$\int\limits_{U_i} T_u f(t)\,du - \int_{U_i^c} T_u f(t)\,du = \int_0^1 f(s) r_i(s)\,ds \cdot r_i(t).$$

This and (20)–(22) yield

$$Q_i(f) = \int_0^1 f(s) r_i(s)\,ds, \quad i = 1, 2, \ldots,$$

i.e., $Q = P$, and Proposition 3 is proved. □

The following result, in fact, is known. However, we provide its proof for completeness.

Lemma 2. *Suppose that the Rademacher sequence is equivalent in a Banach function lattice X on $[0,1]$ to the unit vector basis in l_2, i.e., for some constant $C > 0$ and all $a = (a_k)_{k=1}^\infty \in l_2$*

$$C^{-1}\,\|a\|_{l_2} \le \Big\| \sum_{k=1}^\infty a_k\, r_k \Big\|_X \le C\,\|a\|_{l_2}. \tag{23}$$

Moreover, let $\{r_k\} \subset X'$, where X' is the Köthe dual space for X. Then, the orthogonal projection P is bounded in X if and only if there exists a constant $C_1 > 0$ such that for every $a = (a_k)_{k=1}^\infty \in l_2$

$$\Big\| \sum_{k=1}^\infty a_k\, r_k \Big\|_{X'} \le C_1\,\|a\|_{l_2}. \tag{24}$$

Proof. First, suppose that (24) holds. For arbitrary $f \in X$, we set

$$c_k(f) = \int_0^1 f(s) r_k(s)\,ds, \ \ k = 1, 2, \ldots$$

By (24), for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^n c_k(f)^2 &= \int_0^1 f(s) \sum_{k=1}^n c_k(f) r_k(s)\,ds \le \|f\|_X \Big\| \sum_{k=1}^n c_k(f) r_k \Big\|_{X'} \\ &\le C_1\,\|f\|_X \Big(\sum_{k=1}^n c_k(f)^2 \Big)^{1/2}, \end{aligned}$$

and therefore, taking into account (23), we obtain

$$\|Pf\|_X \le C \Big(\sum_{k=1}^\infty c_k(f)^2 \Big)^{1/2} \le C \cdot C_1 \|f\|_X.$$

Thus, P is bounded in X.

Conversely, if P is a bounded projection in X, then from (23) it follows that

$$\begin{aligned} \int_0^1 f(t) \sum_{k=1}^n a_k r_k(t)\,dt &= \sum_{k=1}^n a_k \cdot c_k(f) \le \|a\|_{l_2} \Big(\sum_{k=1}^n c_k(f)^2 \Big)^{1/2} \\ &\le C\,\|a\|_{l_2}\,\|Pf\|_X \le C\,\|P\|_{X \to X}\,\|a\|_{l_2}\,\|f\|_X \end{aligned}$$

for each $n \in \mathbb{N}$, all $a = (a_k)_{k=1}^{\infty} \in l_2$ and $f \in X$. Hence,

$$\Big\| \sum_{k=1}^{\infty} a_k \, r_k \Big\|_{X'} \le C \, \|P\|_{X \to X} \, \|a\|_{l_2},$$

and (24) is proved. □

Proof of Theorem 3: the case when (10) holds. In view of Lemmas 1, 2 and Proposition 2 it is sufficient to prove that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \Big\| \sum_{k=1}^{n} r_k \Big\|_{(M_{1,w}^d)'} = \infty. \tag{25}$$

For every $m \in \mathbb{N}$ such that $\sqrt{m/2} \in \mathbb{N}$ we consider the set

$$E_m := \{t \in [0,1] : 0 \le \sum_{k=1}^{2m} r_k(t) \le \sqrt{m/2}\,\}.$$

Clearly, $E_m = \cup_{k \in S_m} I_k^{2m}$, where $S_m \subset \{1, 2, \dots, 2^{2m}\}$. Also, it is easy to see that $|E_m| \to 0$ as $m \to \infty$. Denoting

$$f_m := \frac{1}{w(|E_m|)} \, \chi_{E_m}, \;\; m \in \mathbb{N},$$

we show that

$$\|f_m\|_{M_{1,w}^d} \le 1 \;\text{ for all }\; m \in \mathbb{N}. \tag{26}$$

In fact, let I be a dyadic interval from $[0,1]$. Clearly, we can assume that $I \cap E_m \ne \emptyset$. Then, by using the quasi-concavity of w, we have

$$\frac{w(|I|)}{|I|} \int_I |f_m(t)| \, dt = \frac{w(|I|)}{|I|} \cdot \frac{|I \cap E_m|)}{w(|E_m|)} \le \frac{w(|I|)}{|I|} \cdot \frac{|I \cap E_m|}{w(|I \cap E_m|)} \le 1,$$

and (26) is proved.

From (26) it follows that

$$\begin{aligned} \Big\| \sum_{k=1}^{2m} r_k \Big\|_{(M_{1,w}^d)'} \;\; &\ge \;\; \int_0^1 \Big| \sum_{k=1}^{2m} r_k(t) \Big| \cdot f_m(t) \, dt = \frac{1}{w(|E_m|)} \int_{E_m} \Big| \sum_{k=1}^{2m} r_k(t) \Big| \, dt \\ &= \;\; \frac{1}{w(|E_m|)} \, 2^{-2m} \sum_{i \in S_m} \Big| \sum_{k=1}^{2m} \varepsilon_k^i \Big|, \end{aligned}$$

where $\varepsilon_k^i = \text{sign } r_k|_{\Delta_i^{2m}}$, $k = 1, 2, \dots, 2m$, $i \in S_m$. Denoting $\sigma_m := \sum_{i \in S_m} |\sum_{k=1}^{2m} \varepsilon_k^i|$, by the definition of E_m, we obtain

$$\sigma_m = 2 \cdot \sum_{m - \sqrt{m/2} \le k \le m} C_k^{2m} (m-k) = 2 \cdot \sum_{k=1}^{\sqrt{m/2}} C_{m-k}^{2m} \cdot k, \tag{27}$$

where $C_i^n = \frac{n!}{i!(n-i)!}$, $n = 1, 2, \dots$, $i = 0, 1, \dots, n$. Let us estimate the ratio C_{m-k}^{2m}/C_m^{2m} for $1 \le k \le \sqrt{m/2}$ from below. At first,

$$\begin{aligned}\frac{C_{m-k}^{2m}}{C_m^{2m}} &= \frac{(m!)^2}{(m-k)!(m+k)!} = \frac{(m-k+1)\cdot\ldots\cdot(m-1)\cdot m}{(m+1)\cdot\ldots\cdot(m+k-1)\cdot(m+k)} \\ &= \frac{m}{m+k}\cdot\frac{(m-k+1)\cdot\ldots\cdot(m-1)}{(m+1)\cdot\ldots\cdot(m+k-1)} = \frac{m}{m+k}\cdot\prod_{j=1}^{k-1}\frac{1-\frac{j}{m}}{1+\frac{j}{m}} \\ &= \frac{m}{m+k}\cdot\exp\Big(\sum_{j=1}^{k-1}\log\frac{1-\frac{j}{m}}{1+\frac{j}{m}}\Big).\end{aligned}$$

Next, we will need the following elementary inequality

$$\log\frac{1-t}{1+t} + 2\,t + 2\,t^3 \ge 0 \quad \text{for all} \quad 0 \le t \le \frac{1}{2}. \tag{28}$$

Indeed, we set

$$\varphi(t) := \log\frac{1-t}{1+t} + 2\,t + 2\,t^3.$$

Then, $\varphi(0) = 0$. Moreover, for all $t \in [0, 1/2]$ we have

$$\varphi'(t) = -\frac{2}{1-t^2} + 2 + 6t^2 = \frac{2t^2(2-3t^2)}{1-t^2} \ge 0.$$

Thus, $\varphi(t)$ increases on the interval $[0, 1/2]$, and (28) is proved.

From the above formula, inequality (28) and the condition $1 \le k \le \sqrt{m/2}$ we obtain

$$\begin{aligned}\frac{C_{m-k}^{2m}}{C_m^{2m}} &\ge \frac{m}{m+k}\exp\Big(-\frac{2}{m}\sum_{j=1}^{k-1} j - \frac{2}{m^3}\sum_{j=1}^{k-1} j^3\Big) \\ &= \frac{m}{m+k}\exp\Big(\frac{-k(k-1)}{m}\Big)\exp\Big(\frac{-(k-1)^2k^2}{2m^3}\Big) \\ &\ge \frac{1}{2}\exp\Big(-\frac{k^2}{m} - \frac{1}{m}\Big).\end{aligned}$$

Combining this estimate with equality (27), we infer

$$\sigma_m = 2\cdot\sum_{k=1}^{\sqrt{m/2}}\frac{C_{m-k}^{2m}}{C_m^{2m}}\cdot k\cdot C_m^{2m} \ge C_m^{2m}\cdot e^{-1/m}\cdot\sum_{k=1}^{\sqrt{m/2}} e^{-\frac{k^2}{m}}\cdot k. \tag{29}$$

The function $\psi(u) = e^{-\frac{u^2}{m}}\cdot u$ increases on the interval $[0, \sqrt{m/2}]$ because of

$$\psi'(u) = e^{-\frac{u^2}{m}} + ue^{-\frac{u^2}{m}}(-2u/m) = e^{-\frac{u^2}{m}}(1 - 2u^2/m) \ge 0$$

for $0 \le u \le \sqrt{m/2}$. Therefore,

$$\sum_{k=1}^{\sqrt{m/2}} e^{-\frac{k^2}{m}}\cdot k \quad > \quad \sum_{k=1}^{\sqrt{m/2}}\int_{k-1}^{k} e^{-\frac{u^2}{m}}\cdot u\,du = \frac{m}{2}(1 - \frac{1}{\sqrt{e}}) \ge \frac{1}{3}m.$$

Moreover, an easy calculation, by using the Stirling formula, shows that

$$\lim_{m\to\infty} C_m^{2m} 4^{-m}\sqrt{\pi m} = 1.$$

Thus, from the above and (29) it follows that

$$\begin{aligned}\Big\|\sum_{k=1}^{2m} r_k\Big\|_{(M^d_{1,w})'} &\geq \frac{1}{w(|E_m|)}\,2^{-2m}\sum_{i\in S_m}\Big|\sum_{k=1}^{2m}\varepsilon_k^i\Big| = \frac{1}{w(|E_m|)}\,2^{-2m}\,\sigma_m \\ &\geq \frac{1}{w(|E_m|)}\,2^{-2m}\cdot C_m^{2m}\cdot e^{-1/m}\cdot \sum_{k=1}^{\sqrt{m/2}} e^{-\frac{k^2}{m}}\cdot k \\ &\geq \frac{1}{w(|E_m|)}\,4^{-m}\cdot C_m^{2m}\cdot e^{-1/m}\cdot\frac{1}{3}m \approx \frac{\sqrt{m}}{3\sqrt{\pi}\,w(|E_m|)}\end{aligned}$$

for all $m\in\mathbb{N}$ such that $\sqrt{m/2}\in\mathbb{N}$. Since $|E_m|\to 0$, then by (10) $w(|E_m|)\to 0$ as $m\to\infty$. Hence, the preceding inequality implies (25) and the proof is complete. □

5 Structure of Rademacher subspaces in Morrey spaces

Applying Theorem 1 allows us also to study the geometric structure of Rademacher subspaces in Morrey spaces $M_{p,w}$.

THEOREM 4. *Let $1\leq p<\infty$ and $\lim_{t\to 0^+} w(t)=0$. Then every infinite-dimensional subspace of $\mathcal{R}_p$ is either isomorphic to l_2 or contains a subspace, which is isomorphic to c_0 and is complemented in $\mathcal{R}_p$.*

The following two propositions are main tools in the proof of the above theorem.

Proposition 4. *Suppose that $1\leq p<\infty$ and $\lim_{t\to 0^+} w(t)=0$. Then the Rademacher functions form a shrinking basis in $\mathcal{R}_p$.*

Proof. To prove the shrinking property of $\{r_n\}_{n=1}^\infty$ we need to show that for every $\varphi\in(M_{p,w})^*$ we have

$$\|\varphi\big|_{[r_n]_{n=m}^\infty}\|_{(M_{p,w})^*}\to 0 \text{ as } m\to\infty. \tag{30}$$

Assume that (30) does not hold. Then there exist $\varepsilon\in(0,1), \varphi\in(M_{p,w})^*$ with $\|\varphi\|_{(M_{p,w})^*}=1$, and a sequence of functions

$$f_n=\sum_{k=m_n}^\infty a_k^{m_n} r_k, \quad \text{where} \quad m_1<m_2<\dots,$$

such that $\|f_n\|_{M_{p,w}}=1,\ n=1,2,\dots$ and

$$\varphi(f_n)\geq\varepsilon \quad \text{for all} \quad n=1,2,\dots. \tag{31}$$

Let us construct two sequences of positive integers $\{q_i\}_{i=1}^\infty$ and $\{p_i\}_{i=1}^\infty, 1\leq q_1<p_1<q_2<p_2<\dots$ as follows. Setting $q_1=m_1$, we can find $p_1>q_1$, so that $\|\sum_{n=p_1+1}^\infty a_k^{q_1} r_k\|_{M_{p,w}}\leq\varepsilon/2$.

Now, if the numbers $1 \le q_1 < p_1 < q_2 < p_2 < \ldots q_{i-1} < p_{i-1}, i \ge 2$, are chosen, we take for q_i the smallest of numbers m_n, which is larger than p_{i-1} such that

$$w(2^{-q_i}) \le \frac{1}{2}\, w(2^{-q_{i-1}}). \tag{32}$$

Moreover, let $p_i > q_i$ be such that

$$\Big\| \sum_{n=p_i+1}^{\infty} a_k^{q_i} r_k \Big\|_{M_{p,w}} \le \varepsilon/2. \tag{33}$$

We set $\alpha_k^i := a_k^{q_i}$ if $q_i \le k \le p_i$, and $\alpha_k^i := 0$ if $p_i < k < q_{i+1}, i = 1, 2, \ldots$. Then, the sequence

$$u_i := \sum_{k=q_i}^{q_{i+1}-1} \alpha_k^i \, r_k, \quad i = 1, 2, \ldots$$

is a block basis of the Rademacher sequence. Moreover, by the definition of u_i,

$$\sup_{i=1,2,\ldots} \|u_i\|_{M_{p,w}} \le 2, \tag{34}$$

and from the choice of the functional φ and (33) it follows that

$$\varphi(u_i) = \varphi\Big(\sum_{k=q_i}^{p_i} a_k^{q_i}\, r_k \Big) = \varphi(f_i) - \varphi\Big(\sum_{k=p_i+1}^{\infty} a_k^{q_i}\, r_k \Big) \ge \varphi(f_i) - \Big\| \sum_{k=p_i+1}^{\infty} a_k^{q_i}\, r_k \Big\|_{M_{p,w}} \ge \frac{\varepsilon}{2}. \tag{35}$$

Let $\{\gamma_n\}_{n=1}^{\infty}$ be an arbitrary sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \gamma_n^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n = \infty. \tag{36}$$

We show that the series $\sum_{n=1}^{\infty} \gamma_n \, u_n$ converges in $M_{p,w}$. To this end, we set $b_k := \alpha_k^i \cdot \gamma_i$ if $q_i \le k < q_{i+1}$. For every $m \in \mathbb{N}$, by Theorem 1,

$$\Big\| \sum_{n=m}^{\infty} \gamma_n \, u_n \Big\|_{M_{p,w}} = \Big\| \sum_{k=q_m}^{\infty} b_k r_k \Big\|_{M_{p,w}} \approx \Big(\sum_{k=q_m}^{\infty} b_k^2 \Big)^{1/2} + \sup_{l \ge q_m} w(2^{-l}) \cdot \sum_{k=q_m}^{l} |b_k|. \tag{37}$$

Let us estimate both summands from the right-hand side of (37). At first, from (34) and Theorem 1 it follows that

$$\sum_{k=q_m}^{\infty} b_k^2 = \sum_{i=m}^{\infty} \gamma_i^2 \sum_{k=q_i}^{q_{i+1}-1} (\alpha_k^i)^2 \le C_1 \sum_{i=m}^{\infty} \gamma_i^2. \tag{38}$$

Similarly, if $q_m < \ldots < q_{m+r} \le l < q_{m+r+1}$ for some $r = 1, 2, \ldots$, then

$$\begin{aligned} \sum_{k=q_m}^{l} |b_k| \;&=\; \sum_{i=m}^{m+r-1} |\gamma_i| \sum_{k=q_i}^{q_{i+1}-1} |\alpha_k^i| + |\gamma_{m+r}| \sum_{k=q_{m+r}}^{l} |\alpha_k^{m+r}| \\ &\le\; C_2 \Big(\sum_{i=m}^{m+r-1} \frac{|\gamma_i|}{w(2^{-q_{i+1}})} + \frac{|\gamma_{m+r}|}{w(2^{-l})} \Big). \end{aligned}$$

Combining this inequality together with (32), we obtain

$$
\begin{aligned}
w(2^{-l}) \sum_{k=q_m}^{l} |b_k| &\le C_2 \Big(\sum_{i=m}^{m+r-1} |\gamma_i| \frac{w(2^{-q_{m+r}})}{w(2^{-q_{i+1}})} + |\gamma_{m+r}| \Big) \\
&\le C_2 \Big(\sum_{i=m}^{m+r-1} |\gamma_i| \, 2^{-m-r+i+1} + |\gamma_{m+r}| \Big) \\
&\le C_2 \max_{i\ge m} |\gamma_i| \Big(\sum_{j=0}^{r-1} 2^{1+j-r} + 1 \Big) < 3\, C_2 \max_{i\ge m} |\gamma_i|.
\end{aligned}
$$

Clearly, the latter estimate holds also in the simpler case when $q_m \le l < q_{m+1}$. Thus, for every $m \in \mathbb{N}$,

$$
\sup_{l \ge q_m} w(2^{-l}) \sum_{k=q_m}^{l} |b_k| \le 3\, C_2 \max_{i\ge m} |\gamma_i|. \tag{39}
$$

From (36) — (39) it follows that the series $\sum_{n=1}^{\infty} \gamma_n \, u_n$ converges in $M_{p,w}$. At the same time, since $\varphi \in (M_{p,w})^*$, by (35) and (36), we have

$$
\varphi\Big(\sum_{n=1}^{\infty} \gamma_n \, u_n \Big) = \sum_{n=1}^{\infty} \gamma_n \, \varphi(u_n) \ge \frac{\varepsilon}{2} \sum_{n=1}^{\infty} \gamma_n = \infty,
$$

and so (30) is proved. □

Corollary 3. *Under assumptions of Proposition 4:*

(i) $r_k \to 0$ *weakly in* $M_{p,w}$.

(ii) The Rademacher functions form a basis in the dual space $(\mathcal{R}_p)^*$.

Proof. Since $\{r_n\}_{n=1}^{\infty}$ is the biorthogonal system to $\{r_n\}$ itself, (ii) follows from Proposition 4 and Proposition 1.b.1 in [16]. □

Proposition 5. *Let* $1 \le p < \infty$ *and* $\lim_{t\to 0^+} w(t) = 0$. *Suppose that*

$$
u_n = \sum_{k=m_n}^{m_{n+1}-1} a_k \, r_k, \quad 1 = m_1 < m_2 < \dots
$$

is a block basis such that $\|u_n\|_{M_{p,w}} = 1$ *for all* $n \in \mathbb{N}$ *and* $\sum_{k=m_n}^{m_{n+1}-1} a_k^2 \to 0$ *as* $n \to \infty$. *Moreover, let*

$$
w(2^{-m_{n+1}}) \le \frac{1}{2} \, w(2^{-m_n}), \; n = 1, 2, \dots . \tag{40}
$$

Then the sequence $\{u_n\}_{n=1}^{\infty}$ *contains a subsequence equivalent in* $M_{p,w}$ *to the unit vector basis of* c_0.

Proof. Passing to a subsequence if it is needed, without loss of generality we may assume that

$$\sum_{k=m_n}^{m_{n+1}-1} a_k^2 \le \cdot 2^{-n}, \quad n = 1, 2, \ldots. \tag{41}$$

Suppose that $f = \sum_{n=1}^{\infty} \beta_n u_n \in \mathcal{R}_p$. Setting $b_k = a_k \beta_i$ if $m_i \le k < m_{i+1}, i = 1, 2, \ldots$, by Theorem 1, we obtain

$$\|f\|_{M_{p,w}} = \Big\| \sum_{k=1}^{\infty} b_k r_k \Big\|_{M_{p,w}} \approx \Big(\sum_{k=1}^{\infty} b_k^2 \Big)^{1/2} + \sup_{l \in \mathbb{N}} w(2^{-l}) \sum_{k=1}^{l} |b_k|. \tag{42}$$

At first, by (41),

$$\sum_{k=1}^{\infty} b_k^2 = \sum_{i=1}^{\infty} \beta_i^2 \sum_{k=m_i}^{m_{i+1}-1} a_k^2 \le (\sup_{i=1,2,\ldots} |\beta_i|)^2 \cdot \sum_{i=1}^{\infty} 2^{-i} \le \|(\beta_i)\|_{c_0}^2.$$

Moreover, precisely in the same way as in the proof of Proposition 4 from (40) and the equalities $\|u_n\|_{M_{p,w}} = 1, n = 1, 2, \ldots$ it follows that for some constant $C' > 0$

$$\sup_{l=1,2,\ldots} w(2^{-l}) \sum_{k=1}^{l} |b_k| \le C' \, \|(\beta_i)\|_{c_0}.$$

Combining the last two inequalities together with (42), we conclude that $\|f\|_{M_{p,w}} \le C \, \|(\beta_i)\|_{c_0}$ for some constant $C > 0$.

Conversely, since $\{u_n\}$ is an unconditional sequence in $M_{p,w}$ and $\|u_n\|_{M_{p,w}} = 1$, $n = 1, 2, \ldots$, by Theorem 1, $\|f\|_{M_{p,w}} \ge c|\beta_i|$. $i = 1, 2, \ldots$, with some constant $c > 0$. Hence, $\|f\|_{M_{p,w}} \ge c \, \|(\beta_i)\|_{c_0}$, and the proof is complete. $\square$

Proof of Theorem 4. Assume that X is an infinite-dimensional subspace of $\mathcal{R}_p$ such that for every $f = \sum_{k=1}^{\infty} b_k r_k \in X$ we have

$$\|f\|_{M_{p,w}} \approx \Big(\sum_{k=1}^{\infty} b_k^2 \Big)^{1/2},$$

with a constant independent of b_k, $k = 1, 2, \ldots$ Then, X is isomorphic to some subspace of l_2 and so to l_2 itself.

Therefore, if X is not isomorphic to l_2, then there is a sequence $\{f_n\}_{n=1}^{\infty} \subset X, f_n = \sum_{k=1}^{\infty} b_{n,k} r_k$, such that $\|f_n\|_{M_{p,w}} = 1$ and

$$\sum_{k=1}^{\infty} b_{n,k}^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{43}$$

Observe that $\{f_n\}_{n=1}^{\infty}$ does not contain any subsequence converging in $M_{p,w}$-norm. In fact, if $\|f_{n_k} - f\|_{M_{p,w}} \to 0$ for some $\{f_{n_k}\} \subset \{f_n\}$ and $f \in X$, then from Theorem 1 and (43) it follows that $f = \sum_{k=1}^{\infty} b_k r_k$, where $b_k = 0$ for all $k = 1, 2, \ldots$. Hence, $f = 0$. On the other hand, obviously, $\|f\|_{M_{p,w}} = 1$, and we come to a contradiction.

Thus, passing if it is needed to a subsequence, we can assume that

$$\|f_n - f_m\|_{M_{p,w}} \geq \varepsilon > 0 \ \text{ for all } \ n \neq m. \tag{44}$$

Recall that, by Corollary 3, the sequence $\{r_k\}_{k=1}^\infty$ is a basis of the space $(\mathcal{R}_p)^*$. Applying the diagonal process, we can find the sequence $\{n_k\}_{k=1}^\infty, n_1 < n_2 < \ldots$, such that for every $i = 1, 2, \ldots$ there exists $\lim_{k\to\infty} \int_0^1 r_i(s) f_{n_k}(s)\, ds$. Then,

$$\lim_{k\to\infty} \int_0^1 r_i(s)(f_{n_{2k+1}}(s) - f_{n_{2k}}(s))\, ds = 0 \ \text{ for all } \ i = 1, 2, \ldots.$$

Hence, since the sequence $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty$ is bounded in $M_{p,w}$ we infer that $f_{n_{2k+1}} - f_{n_{2k}} \to 0$ weakly in $M_{p,w}$. Now, taking into account (44) and applying the well-known Bessaga-Pełczyński Selection Principle (cf. [1, Proposition 1.3.10, p. 14]), we may construct a subsequence of the sequence $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty$ (we keep for it the same notation) and a block basis

$$u_k = \sum_{j=m_k}^{m_{k+1}-1} a_j r_j, \ 1 = m_1 < m_2 < \ldots,$$

such that

$$\|u_k - \big(f_{n_{2k+1}} - f_{n_{2k}}\big)\|_{M_{p,w}} \leq B_0^{-1} \cdot 2^{-k-1}, \quad k = 1, 2, \ldots, \tag{45}$$

where B_0 is the basis constant of $\{r_k\}$ in $\mathcal{R}_p$, and

$$w(2^{-m_{k+1}}) \leq \frac{1}{2} \cdot w(2^{-m_k}), \quad k = 1, 2, \ldots. \tag{46}$$

From (45) it follows that the sequences $\{u_k\}_{k=1}^\infty$ and $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty$ are equivalent in $M_{p,w}$ (cf. [16, Proposition 1.a.9]). Moreover, by Theorem 1 and (43),

$$\sum_{j=m_k}^{m_{k+1}-1} a_j^2 \to 0 \ \text{ as } \ k \to \infty.$$

This fact together with inequality (46) allows us to apply Proposition 5, which implies that the sequence $\{u_k\}_{k=1}^\infty$ (and so $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty$) contains a subsequence equivalent to the unit vector basis of c_0. Since $\{f_{n_{2k+1}} - f_{n_{2k}}\}_{k=1}^\infty \subset X$, then X contains a subspace isomorphic to c_0. Complementability of this subspace in $\mathcal{R}_p$ is an immediate consequence of Sobczyk's theorem (see [1, Corollary 2.5.9]). □

Remark 2. If $\lim_{t\to 0^+} w(t) > 0$, then $M_{p,w} = L_\infty$ and $\{r_k\}$ is equivalent in $M_{p,w}$ to the unit vector basis of l_1 (cf. Theorem 1). Observe also that if $\sup_{0<t\leq 1} w(t) \log_2^{1/2}(2/t) < \infty$, then we get another trivial situation: $\mathcal{R}_p \simeq l_2$ (see Corollary 2).

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